A generalized Tullock contest

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Abstract

We construct a generalized Tullock contest under complete information where contingent upon winning or losing, the payoff of a player is a linear function of prizes, own effort, and the effort of the rival. This structure nests a number of existing contests in the literature and can be used to analyze new types of contests. We characterize the unique symmetric equilibrium and show that small parameter modifications may lead to substantially different types of contests and hence different equilibrium effort levels.

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1. Introduction

Contests are economic or social interactions in which two or more players expend costly resources in order to win a prize. The resources expended by players determine their probability of winning a prize. In this article we construct a generalized Tullock contest under complete information. We consider a simple two-player contest where, contingent upon winning or losing, a player receives different prizes. Players' outcome-contingent payoffs are linear functions of prizes, own effort, and the effort of the rival.[1] This structure nests a number of existing contests in the literature and can be used to analyze new types of contests. We characterize the unique symmetric equilibrium and show that small parameter modifications may lead to substantially different types of contests and hence different equilibrium effort levels.

The rent-seeking contest literature originated with Tullock (1980). In this model, player i's probability of winning is $p_i(x_i, x_j) = x_i^r/(x_i^r + x_j^r)$, where x_i and x_j are the efforts of players i and j. The function, $p_i(x_i, x_j)$, that maps efforts into probabilities of winning is called the contest success function (CSF). The most popular versions of the Tullock CSF are the lottery (r = 1) and the all-pay auction ($r = \infty$).[2] There are several reasons why Tullock's CSF is widely employed. First, a number of studies have provided axiomatic justification for it (Skaperdas 1996; Clark and Riis 1998). Second, Baye and Hoppe (2003) have identified conditions under which a variety of rent-seeking contests, innovation tournaments, and patent-race games are strategically equivalent to the Tullock contest.

Economists often use modified payoffs in the Tullock contest in order to address specific research questions. For example, Skaperdas and Gan (1995) restrict the losing payoff to study the effect of risk aversion in a "limited liability" contest. Cohen and Sela (2005) restrict the winning payoff to show that in certain contests a weaker contestant can win with higher probability than a stronger contestant. Many other studies use modified payoffs in the Tullock contest, a short list of example includes Chung (1996), Alexeev and Leitzel (1996), Lee and Kang (1998), Amegashie (1999), Glazer and Konrad (1999), Garfinkel and Skaperdas (2000), Grossman and Mendoza (2001), Öncüler and Croson (2005), and Matros and Armanios (2009).

In this article we propose a generalized Tullock contest in which payoffs are linear functions of prizes, own effort, and the effort of the rival. Our model nests a number of the existing contests in the literature and also provides a framework for studying new contests. One of the main motivations for introducing a generalized structure is the fact that in many real life contests payoffs are endogenous, i.e., payoffs depend both on the individual and on the rival's effort. For example, in innovation contests one firm's R&D effort may provide information spillovers that benefit its rival (D'Aspremont and Jacquemin 1988; Kamien et al. 1992). In a patent race the expenditure of a rival can decrease the patent value for the winner, creating a negative spillover (Alexeev and Leitzel 1996). Negative spillovers are often observed in military conflicts between countries (Garfinkel and Skaperdas 2000) or in biological survival contests (Baker 1996). Another example where spillovers are important is litigation (Farmer and Pecorino 1999; Baye et al. 2005). Depending on the litigation system, losers have to compensate winners for a portion of their legal expenditures or up to the amount actually spent by the loser. These create either negative or positive spillover effects of one party's expenditure on another. Baye et al. (2010) model the spillovers in terms of an all-pay auction contest. We explicitly model such spillovers in the context of a Tullock lottery contest.[3]

2. Theoretical model

We consider a two-player contest with two prizes. The players are denoted by i and j. Both players value the winning prize as $W > 0$ and the losing prize as $L \in \mathbb{R}$. We assume that winning the prize provides higher valuation than losing, i.e., $W > L$. Players simultaneously expend irreversible and costly efforts $x_i \geq 0$ and $x_j \geq 0$. The probability of player i winning the contest is described by a Tullock lottery CSF:

$$p_i(x_i, x_j) = \begin{cases} x_i/(x_i + x_j) & \text{if } x_i + x_j \neq 0 \\ 1/2 & \text{if } x_i = x_j = 0 \end{cases} \tag{1}$$

Contingent upon winning or losing, the payoff for player i is a linear function of prizes, own effort, and the effort of the rival:

$$\pi_i(x_i, x_j) = \begin{cases} W + \alpha_1 x_i + \beta_1 x_j & \text{with probability} \quad p_i(x_i, x_j) \\ L + \alpha_2 x_i + \beta_2 x_j & \text{with probability} \ 1 - p_i(x_i, x_j) \end{cases} \tag{2}$$

where α_1, α_2 are cost parameters, and β_1, β_2 are spillover parameters. To ensure that a player has no incentive to expend infinite effort, we impose conditions that a player's own effort has a negative direct impact on his winning payoff and a non-positive direct impact on his losing payoff, that is, $\alpha_1 < 0$ and $\alpha_2 \leq 0$.

We define the contest described by (1) and (2) as $\Gamma(i, j, \Omega)$, where $\Omega = \{W, L, \alpha_1, \alpha_2, \beta_1, \beta_2\}$ is the parameter space. All parameters in Ω along with the CSF are common knowledge for both players. The players are assumed to be risk neutral; therefore, for a given effort pair (x_i, x_j), the expected payoff for player i in contest $\Gamma(i, j, \Omega)$ is:

$$E\big(\pi_i(x_i, x_j)\big) = \frac{x_i}{x_i + x_j}\big(W + \alpha_1 x_i + \beta_1 x_j\big) + \frac{x_j}{x_i + x_j}\big(L + \alpha_2 x_i + \beta_2 x_j\big) \tag{3}$$

where $(x_i, x_j) \neq (0,0)$. For $x_i = x_j = 0$, the expected payoff is $E\big(\pi_i(x_i, x_j)\big) = (W + L)/2$. By setting $A = W - L + (\beta_1 - \beta_2)x_j$, $B = \alpha_1 - \alpha_2$, and $C = L + \beta_2 x_j$, expression (3) can be rewritten as:

$$E\big(\pi_i(x_i, x_j)\big) = B\frac{x_i^2}{x_i + x_j} + A\frac{x_i}{x_i + x_j} + \alpha_2 x_i + C \tag{4}$$

Player i's best response is derived by maximizing $E\big(\pi_i(x_i, x_j)\big)$ with respect to x_i. Differentiating equation (4) with respect to x_i yields the following first order condition:

$$\frac{dE\big(\pi_i(x_i, x_j)\big)}{dx_i} = B\frac{x_i^2 + 2x_i x_j}{(x_i + x_j)^2} + A\frac{x_j}{(x_i + x_j)^2} + \alpha_2 \tag{5}$$

The second order condition is:

$$\frac{d^2E\big(\pi_i(x_i, x_j)\big)}{dx_i^2} = (Bx_j - A)\frac{2x_j}{(x_i + x_j)^3} \tag{6}$$

From the second order condition (6) it is easy to verify that the payoff function for player i is concave as long as:

$$x_j \leq \frac{W - L}{(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)} \tag{7}$$

If (7) holds then first order condition is necessary and sufficient for maximizing player i's payoff. Consequently by solving (5) for x_i and by substituting back the values of A and B, we receive the best response function of x_i in terms of the effort choice of x_j:

$$x_i^{BRF} = -x_j + \sqrt{\frac{\{(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)\}x_j^2 - \{W - L\}x_j}{\alpha_1}} \tag{8}$$

if $x_j \leq (W - L)/(-\alpha_2 - \beta_1 + \beta_2)$; and $x_i^{BRF} = 0$, otherwise.[4] It is clear that the best response function (8) depends on α_1, α_2, the difference between β_1 and β_2, and the spread between the winning and the losing prize valuations.

By simultaneously solving best response functions (8), and accounting for symmetric Nash equilibrium we obtain the unique equilibrium in which player i and j expend efforts of

$$x_i^* = x_j^* = x = \frac{(W-L)}{-(3\alpha_1+\alpha_2)-(\beta_1-\beta_2)} \tag{9}$$

The expected equilibrium payoff in the symmetric equilibrium is given by:

$$E^*(\pi) = \frac{(\beta_2-\alpha_1)(W-L)}{-(3\alpha_1+\alpha_2)-(\beta_1-\beta_2)} + L \tag{10}$$

Both the non-negative equilibrium effort condition and the second order condition hold if $-(3\alpha_1 + \alpha_2) - (\beta_1 - \beta_2) > 0$. Furthermore, to ensure that both players are willing to expend positive efforts in equilibrium the equilibrium payoff has to be greater than or equal to the payoff of losing, i.e., $E^*(\pi) \geq L$. This condition translates into $\beta_2 - \alpha_1 \geq 0$ and it means that the unit cost of winning has to be lower than the unit spillover benefit from losing.

3. Existing contests in the literature

3.1. Contests without spillovers

In the standard contest defined by Tullock (1980), both players have the same valuation for the prize and despite the outcome of the contest the efforts of both players are lost. In such a case, $W > 0$, $\alpha_1 = \alpha_2 = -1$, and the other parameters in Ω are zero. The best response function for player i is $x_i = -x_j + \sqrt{Wx_j}$ (Figure 1). The unique equilibrium is the symmetric equilibrium with $x_i^* = x_j^* = W/4$.

[Figure 1 is about here]

Skaperdas and Gan (1995) examine a 'limited liability' case in which the loser's payoff is independent of the efforts expended. The authors motivate this example by stating that contestants may be entrepreneurs who borrow money to spend on research and development and thus are not legally responsible in case of loss. The loser of such a contest is unable to repay the loan and goes bankrupt. In such a case, $W > 0$, $\alpha_1 = -1$, and the other parameters in Ω are zero.

The best response function for player i is $x_i = -x_j + \sqrt{x_j^2 + Wx_j}$ (Figure 1). Under the symmetric equilibrium we have $x_i^* = x_j^* = W/3$.

Garfinkel and Skaperdas (2000) consider a case in which two players compete to win a war. In this game player i and j have resource endowments of V_i and V_j which they can use to win the contest. The winner receives the sum of resources minus the sum of efforts expended by both players. It is also assumed that war destroys a fraction $(1-\phi) \in (0,1)$ of the total payoff. Thus, the needed restrictions are $W = \phi(V_i + V_j)$, $\alpha_1 = \beta_1 = -\phi$, and the other parameters in Ω are zero. The best response function is $x_i = -x_j + \sqrt{(V_i + V_j)x_j}$ (Figure 1, where $V_i + V_j = 2W$). Although V_i and V_j can be different, the equilibrium efforts for players i and j are the same, i.e., $x_i^* = x_j^* = (V_i + V_j)/4$.

3.2. Contests with spillovers

A simple linear version of the Chung (1996) contest with positive spillovers can be captured by $\Gamma(i, j, \{W, 0, a-1, -1, a, 0\})$, where $a \in (0,1)$ is the degree of spillover. The corresponding best response function is $x_i = -x_j + \sqrt{Wx_j/(1-a)}$and the symmetric equilibrium efforts are $x_i^* = x_j^* = W/[4(1-a)]$. Similarly, a contest of Alexeev and Leitzel (1996), where the value of the winning prize decreases with the total effort expenditures, can be captured by $\Gamma(i, j, \{W, 0, -1, -1, -1, 0\}$. The resulting best response function is $x_i = -x_j + \sqrt{Wx_j - x_j^2}$ and the symmetric equilibrium efforts are $x_i^* = x_j^* = W/5$.

Baye et al. (2005) examine and compare several litigation systems under the all-pay auction CSF. We use the Tullock lottery CSF in Baye et al. (2005) structure by restricting $L = 0$,

$\alpha_1 = -\beta$, $\beta_1 = -(1-\alpha)$, $\alpha_2 = -\alpha$, and $\beta_2 = -(1-\beta)$, where $\alpha \in (0,1)$ and $\beta \in (0,1)$. Interestingly enough, when we restrict the parameters to match their model, the best response function $x_i = -x_j + \sqrt{W x_j/\beta}$ is independent of the value of α. Note that when $\beta = 1$ (i.e., the case of American, Marshall, and Quayle systems of litigation), the best response function as well as the symmetric equilibrium turns out to be qualitatively equivalent to that in Tullock (1980).

Similarly, the two-player versions of other contests by Farmer and Pecorino (1998), Lee and Kang (1998), Amegashie (1999), Glazer and Konrad (1999), Garfinkel and Skaperdas (2000), Grossman and Mendoza (2001), and Matros and Armanios (2009) can be obtained from our generalized contest by placing appropriate parameter restrictions.

4. New contests

4.1. Contests without spillovers

In a standard Tullock contest the unit cost of losing is the same as the unit cost of winning. However, in many real life situations we observe that the winner of the contest pays less than the loser. A prominent example is the government procurement auction for defense weapons. Different companies make costly investments to produce prototypes and the government shares the prototype's production cost with only the winner.[5] In these cases, the winner of the contest faces lower marginal cost than the loser. Rightfully, this contest can be called a 'lazy winner' contest. We can capture this by setting $W > 0$, $\alpha_2 < \alpha_1 < 0$ and other parameters in Ω to zero. Therefore, the payoff for player i is given by

$$\pi_i(x_i, x_j) = \begin{cases} W + \alpha_1 x_i & \text{with probability} \quad p_i(x_i, x_j) \\ \alpha_2 x_i & \text{with probability} \quad 1 - p_i(x_i, x_j) \end{cases} \tag{11}$$

The resulting best response function is $x_i = -x_j + \sqrt{\{(\alpha_1 - \alpha_2)x_j^2 - Wx_j\}/\alpha_1}$ and the symmetric equilibrium effort levels are $x_i^* = x_j^* = W/(-3\alpha_1 - \alpha_2)$.

4.2. Contests with spillovers

Next, we consider an 'input spillover' contest where the effort expended by player *j* *partially* benefits player i and vice versa. This case can be interpreted as the input spillover effect in R&D innovation (Kamien et al., 1992). In our model we assume that the winner (loser) of the contest receives a benefit proportional to the loser's (winner's) effort. After setting $\alpha_1 = \alpha_2 = -1$, and $L = 0$ the payoff function of 'input spillover' contest takes the form:

$$\pi_i(x_i, x_j) = \begin{cases} W - x_i + \beta_1 x_j & \text{with probability } p_i(x_i, x_j) \\ -x_i + \beta_2 x_j & \text{with probability } 1 - p_i(x_i, x_j) \end{cases} \tag{12}$$

where $\beta_1 \geq 0$, $\beta_2 \geq 0$, and $\beta_1 - \beta_2 < 4$.

[Figure 2 is about here]

Note that the best response function, $x_i = -x_j + \sqrt{(\beta_1 - \beta_2)x_j^2 + Wx_j}$, changes dramatically with β_1 and β_2. The symmetric equilibrium effort of this contest is given by $x_i^* = x_j^* = W/(4 - \beta_1 + \beta_2)$. Hence, a player expends more (less) effort with an increase in the spillover benefit from winning (losing). Figure 2 displays best response functions and resulting equilibria for different values of β_1 and β_2. As we move left to right, $(\beta_1 - \beta_2)$ decreases, and the total effort expended also decreases. This has a simple intuition: if the positive externality gained by losing increases relative to that of winning then the players will spend less effort to win the contest. This case resembles R&D contests in countries where property rights are not protected by the government and the spillover in case of losing is very large. Therefore, there is a strong incentive to free ride on the effort of the others.

5. Discussion

In this article we construct a generalized Tullock contest under complete information. We show how different existing contests in the literature can be nested under this generalized structure. We also characterize the unique symmetric equilibrium and show that small parameter modifications may lead to substantially different equilibrium effort levels. Finally, we introduce and characterize two new contests to the literature. Our results can be applied to the fields of labor economics, law and economics, industrial organization, public economics, and political economy. By applying certain parameter restrictions to our model one can also imitate the rent-seeking contests, patent races, military combats, or legal conflicts.

There are a number of interesting extensions of our analysis. For example, one can use our generalized structure to meet a given objective of a contest designer. This objective varies between contests. In sports or social benefit programs the designer may want to maximize the total expenditures of effort, whereas in rent-seeking or electoral contests the designer may want to minimize them. For a given objective, one can appropriately set the parameters of our model so that the desired outcome is achieved. Other extensions include contests with more than two players, the effects of risk aversion and incomplete information. Finally, it would be interesting to test empirically the predictions of our generalized contest model. In particular, our analysis demonstrates that small parameter modifications may lead to substantially different equilibrium effort levels. To test these predictions, one could design an experiment similar to Sheremeta (2010a, 2010b). We leave these questions for future research.

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Endnotes

[1] Contests are characterized by three attributes such as prizes, players, and the efforts of the players (Konrad 2009).

[2] In a first-price all-pay auction the winner is the player who expends the most effort (Baye et al. 1996).

[3] Chung (1996) is among the first to consider spillover/externality of rival's efforts in a contest framework. Our generalized model differs substantially from Chung's model. First, Chung (1996) uses strictly non-linear spillovers, whereas the current model considers linear spillovers. Second, Chung's model incorporates strictly endogenous prizes and strictly positive spillovers from winning (i.e., the winning prize is a strictly increasing and concave function of the *total* effort), whereas the current model captures both positive and negative spillovers from winning and from losing. Moreover, the current model captures the cases where the prizes are exogenous, or a function of only one of the player's efforts. We thank an anonymous referee for pointing out the differences.

[4] Note that the restriction (7) is weaker than the restriction needed for (8) to be well defined. Hence, when the best response is positive then solving the best response functions will lead us to an equilibrium.

[5] See Kaplan et al. (2002) for a detailed discussion. Matros and Armanios (2009) also study a very similar contest.

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Figures

Figure 1 – Best response functions and resulting equilibria (*W* = 1)

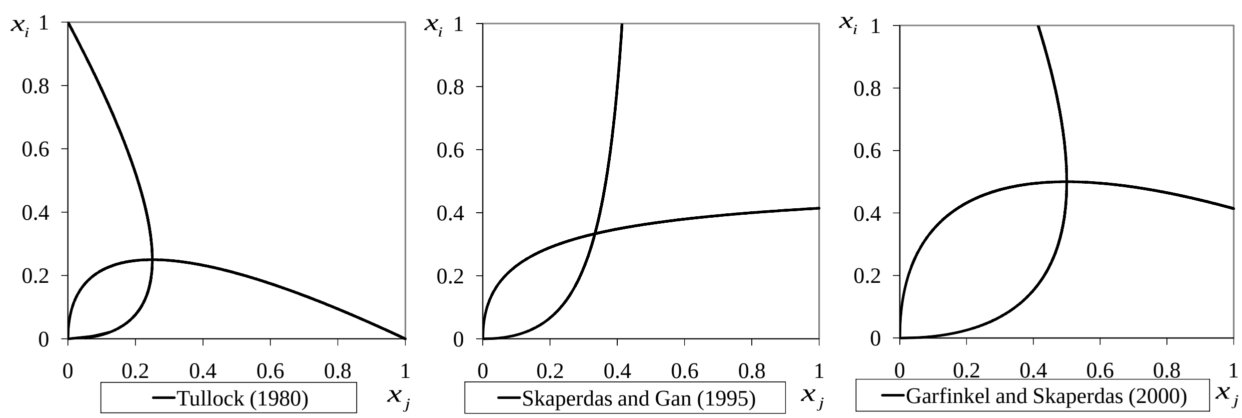

Figure 2 – Best response functions for 'input spillover' contest (*W* = 1)